Threshold expansion of the sunset diagram

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Abstract

By use of the threshold expansion we develop an algorithm for analytical evaluation, within dimensional regularization, of arbitrary terms in the expansion of the (two-loop) sunset diagram with general masses m_1, m_2 and m_3 near its threshold, i.e. in any given order in the difference between the external momentum squared and its threshold value, $(m_1 + m_2 + m_3)^2$. In particular, this algorithm includes an explicit recurrence procedure to analytically calculate sunset diagrams with arbitrary integer powers of propagators at the threshold.

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1. Introduction

The purpose of our paper is to apply explicit prescriptions [1] recently obtained for the expansion near threshold, i.e. in powers of the difference between the external momentum squared and its threshold value, to calculation of the sunset diagram (see Fig. 1) with general masses and powers of propagators. We thereby extend, to the threshold expansion, analysis of two-loop Feynman integrals performed in a series of papers [2–5] and based on explicit formulae for the asymptotic expansion of Feynman diagrams in various off-shell limits of momenta and masses [6] (see also earlier papers [7] and operator analogues in [8]; for brief reviews, see in [9]), as well as for some typically Minkowskian on-shell limits [10]. In particular, in ref. [3] the threshold behaviour at small (as compared to some masses not involved in the cut) non-zero thresholds was examined within the large mass expansion, and the description of three-particle thresholds depended crucially on a possibility to describe the threshold behaviour of the sunset diagram and a similar diagram with four propagators.

The sunset diagram (Fig. 1) represents the simplest example of a diagram involving a three-particle threshold, at $k^2 = (m_1 + m_2 + m_3)^2$. When one or two internal particles are massless, the (four-dimensional) results can be obtained in terms of dilogarithms [11,12][3]. The situation gets more complicated when all three virtual particles involved in the cut are massive. Such a situation occurs e.g. in the two-loop off-shell contributions to the Higgs self energy. Although in three dimensions the result for this diagram (even with different masses) is quite simple [14] (cf. also in ref. [15]), in four dimensions no exact results in terms of known functions (like polylogarithms, etc.) are available in this (totally massive) case. Moreover, there are arguments [16] that the result cannot be expressed in terms of polylogarithms, except for the special cases like the threshold (and pseudothresholds).

One possibility is to use various integral representations [12, 17–19], in order to get numerical values for given masses and the momentum. Another possibility is to study analytic expansions in different regions. For instance, using the algorithms presented in [2], one can construct several terms of the small momentum expansion and the large momentum expansion. Moreover, in ref. [20] a closed formula for the corresponding coefficients (in the case of the sunset diagram) was derived, and the occurring hypergeometric series were identified as Lauricella functions of three variables. Nevertheless, because of difficulties in constructing analytic continuation of these functions, one cannot extract much information about the threshold behaviour. We also note that in ref. [21] a differential equation for the sunset diagram was constructed, whose derivation was essentially based on the integration by parts [22] and a recurrence procedure described in [23]. The first step towards explicit evaluation of the coefficients of the threshold expansion was done in ref. [24], where the threshold (and pseudothreshold) values of the sunset diagram (with unit powers of the propagators, and also in the case when one of the powers is equal to two) were calculated analytically in terms of dilogarithms of the mass ratios.

The sunset diagram with general powers of propagators and masses is given by the

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[3]Some other results for two-loop two-point diagrams with masses can be found in refs. [13].

following two-loop integral (cf. Fig. 1):

$$L(n; \nu_1, \nu_2, \nu_3) \equiv \int \int \frac{\mathrm{d}^n p_1 \, \mathrm{d}^n p_2}{[p_1^2 - m_1^2]^{\nu_1} \, [p_2^2 - m_2^2]^{\nu_2} \, [(k - p_1 - p_2)^2 - m_3^2]^{\nu_3}} \, , \quad (1)$$

where $n = 4 - 2\varepsilon$ is the space-time dimension, in the framework of dimensional regularization [25]. We shall use the notation

$$\mu_T \equiv m_1 + m_2 + m_3 \, , \qquad \xi_i \equiv \frac{m_i}{\mu_T}, \quad (2)$$

with $\sum_{i=1}^3 \xi_i = 1$, and introduce the expansion parameter

$$y \equiv \mu_T^2 - k^2 \, . \quad (3)$$

To make the dependence on y manifest, it is convenient to shift the loop momenta in eq. (1) as $p_1 \rightarrow p_1 + \xi_1 k$, $p_2 \rightarrow p_2 + \xi_2 k$. Thus the r.h.s. of eq. (1) becomes

$$\int \int \frac{\mathrm{d}^n p_1 \, \mathrm{d}^n p_2}{[p_1^2 + 2\xi_1(k p_1) - \xi_1^2 y]^{\nu_1} \, [p_2^2 + 2\xi_2(k p_2) - \xi_2^2 y]^{\nu_2} \, [(p_1 + p_2)^2 - 2\xi_3(k, p_1 + p_2) - \xi_3^2 y]^{\nu_3}} \, . \quad (4)$$

Without loss of generality, in the time-like region we can choose a frame with the external momentum $k = (k_0, \vec{0})$ with $k_0 > 0$. According to the general prescriptions of the threshold expansion [1], one should consider every loop momentum p_i ($i = 1, 2$) to be of the following four types:

$$\textit{hard} \text{ (h): } p_{i0} \sim |k|, \; |\vec{p}_i| \sim |k|,$$

$$\textit{soft} \text{ (s): } p_{i0} \sim \sqrt{y}, \; |\vec{p}_i| \sim \sqrt{y},$$

$$\textit{potential} \text{ (p): } p_{i0} \sim y/|k|, \; |\vec{p}_i| \sim \sqrt{y},$$

$$\textit{ultrasoft} \text{ (us): } p_{i0} \sim y/|k|, \; |\vec{p}_i| \sim y/|k| \, ,$$

where $|k| \equiv \sqrt{k^2} = k_0$.

We shall consider the most complicated case when all three masses are non-zero. Then only (h-h) and (p-p) regions contribute to the threshold expansion because, for any other region, one obtains scaleless integrals which are naturally put to zero (they are analogous to massless tadpoles in dimensional regularization [25]). Note that if one or two masses were zero then we would have non-zero (us-p) or (us-us) contributions, instead of the (p-p) one (cf. in refs. [1,26]).

As the main example, we shall treat the diagram with $\nu_1 = \nu_2 = \nu_3 = 1$. Within threshold expansion, this "master" sunset diagram is represented as

$$L(4 - 2\varepsilon; 1, 1, 1) = -\pi^{4-2\varepsilon} \, \Gamma(1 + 2\varepsilon) \, k^2 \sum_{j=0}^{\infty} \left(\frac{y}{k^2} \right)^j \left[C_j^{(\mathrm{h-h})} (k^2)^{-2\varepsilon} + C_j^{(\mathrm{p-p})} y^{-2\varepsilon} \right] \, , \quad (5)$$

where we have extracted the factor $i\pi^{n/2}$ per loop. Our results can be easily generalized for sunset diagrams with any indices ν_i.

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The (h-h) contribution is given by expanding the integrand in (4) in Taylor series in y, so that these are sunset integrals, with various (higher) indices of the lines, evaluated exactly at the threshold ($y = 0$). In sections 2–4 we shall describe how an arbitrary integral of this kind can be analytically evaluated. It happens that general solutions of recurrence relations constructed in [23] cannot be directly applied at the threshold. Nevertheless, those relations are useful at the threshold, providing equations which should be used together with the usual integration-by-parts relations [22].

The (p-p) contribution is given by expanding the propagators in (4) in Taylor series in the squares of time components, p_{10}^2, p_{20}^2 and $(p_{10} + p_{20})^2$. The evaluation of the (p-p) part of the expansion is presented in Section 5. In Section 6 we compare our results for the threshold expansion with results based on numerical integration. We briefly summarize our results in Section 7.

In general, the threshold expansion has a complicated structure and it is difficult to see how it works. Nevertheless, in our particular example we can present some simple arguments. Namely, it happens that the remainder of the expansion of the sunset diagram can be constructed by applying, to eq. (4), an operator

$$\mathcal{R}^{(N)} = \left(1 - \mathcal{T}_y^{(N)}\right) \left(1 - \mathcal{T}_{p_{i0}^2}^{(N-2)}\right). \tag{6}$$

Here $\mathcal{T}_y^{(N)}$ denotes an operator that picks up the first N terms of Taylor expansion in y (cf. eq. (8) below), whereas $\mathcal{T}_{p_{i0}^2}^{(N)}$ corresponds to the first N terms of the expansion in squares of time components p_{10}^2, p_{20}^2 and $(p_{10}+p_{20})^2$ (cf. eq. (47) below). It is assumed that these operators "commute" with the the loop integrations, i.e. they should be applied directly to the denominators in eq. (4).

For instance, for the integral with $\nu_i = 1$ we can write

$$L(n; 1, 1, 1) = \left(1 - \mathcal{R}^{(N)}\right) L(n; 1, 1, 1) + \mathcal{R}^{(N)} L(n; 1, 1, 1)$$

$$= \left(\mathcal{T}_y^{(N)} + \mathcal{T}_{p_{i0}^2}^{(N-2)} - \mathcal{T}_y^{(N)} \mathcal{T}_{p_{i0}^2}^{(N-2)}\right) L(n; 1, 1, 1) + \mathcal{R}^{(N)} L(n; 1, 1, 1), \tag{7}$$

where the terms $\mathcal{T}_y^{(N)} L$ and $\mathcal{T}_{p_{i0}^2}^{(N-2)} L$ give us the (h-h) and (p-p) contributions to the threshold expansion (5), respectively. For dimensionally-regularized integrals, the product of these operators, $\mathcal{T}_y^{(N)} \mathcal{T}_{p_{i0}^2}^{(N-2)} L$, vanishes. To see this, one can integrate in p_{10} and p_{20} using residual theorem: then, the resulting integrals in $\vec{p}_1$ and $\vec{p}_2$ are scaleless (i.e. analogous to massless tadpoles, but in $(n - 1)$ dimensions), and therefore vanish.

Finally, the last term on the r.h.s. of eq. (7), $\mathcal{R}^{(N)} L(n; 1, 1, 1)$, plays the role of the remainder of the expansion. To show this, consider first the region of small loop momenta where the operator $\mathcal{T}_y^{(N)}$ is dangerous because it produces infrared threshold singularities. However these singularities are removed by the operator $\left(1 - \mathcal{T}_{p_{i0}^2}^{(N-2)}\right)$ which increases powers of the variables p_1 and p_2 in the numerator. Then, consider the region of large loop momenta where the operator $\mathcal{T}_{p_{i0}^2}^{(N-2)}$ is dangerous because it generates specific divergences (corresponding to ultraviolet singularities in lower dimensions). Still, the

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latter are removed by the operator $\left(1 - \mathcal{T}_y^{(N)}\right)$ which effectively increases the powers of integration momenta in the denominators. Thus the remainder $\mathcal{R}^{(N)}$ does not involve new divergences, neither ultraviolet nor infrared ones, as compared to the initial Feynman integral (4). Moreover, taking into account the operator $\left(1 - \mathcal{T}_y^{(N)}\right)$ we see that the order of this remainder is y^{N+1}, up to logarithms.

2. Taylor expansion and the (h-h) contribution

The formal expansion in y of the denominators in the integrand of eq. (4) can be performed via

$$\frac{1}{[p_i^2 + 2\xi_i(kp_i) - \xi_i^2 y]^{\nu_i}} = \sum_{j_i=0}^{\infty} \frac{(\nu_i)_{j_i}}{j_i!} \frac{\xi_i^{2j_i} y^{j_i}}{[p_i^2 + 2\xi_i(kp_i)]^{\nu_i+j_i}} \, , \tag{8}$$

where $(\nu)_j \equiv \Gamma(\nu+j)/\Gamma(\nu)$ is the Pochhammer symbol. If we denote $p_3 = -p_1 - p_2$, the third denominator can also be taken into account by eq. (8). Then, collecting the terms with given powers of y, we get the (h-h) contributions (cf. eq. (5)). According to the r.h.s. of eq. (8), each separate term corresponds to the integral (4) with shifted powers of propagators $(\nu_i \to \nu_i + j_i)$ and $y = 0$, i.e. at the threshold.

One should however keep in mind which variables are considered as independent ones. Let us consider ξ_i, eq. (2), as "external" parameters characterizing the mass ratios. Then the remaining variables are y, k^2 and $\mu_T^2 = (m_1 + m_2 + m_3)^2$, which are dependent, according to the definition (3). Since y is the main parameter of the threshold expansion (cf. eq. (5)), we can choose between the following pairs of independent variables: (i) y and k^2, and (ii) y and μ_T^2.

On one hand, in the context of the threshold expansion [1] (including the related issue of the non-relativistic limit), the first set, y and k^2, looks more natural (cf. eq. (5)). In particular, the momentum integrals arising after expansion in y (8) are functions of k^2. In section 5, we shall also see that such a choice is more convenient for evaluating the (p-p) contribution. On the other hand, when evaluating the threshold values of the momentum-space integrals (corresponding to the (h-h) contribution), it is convenient to put $k^2 = \mu_T^2$, i.e. to calculate

$$L_T(n; \nu_1, \nu_2, \nu_3) \equiv L(n; \nu_1, \nu_2, \nu_3)|_{k^2=(m_1+m_2+m_3)^2} \, . \tag{9}$$

This would correspond to the second set of independent variables, y and μ_T^2. In particular, when $\nu_i = 1$ we define

$$L(4 - 2\varepsilon; 1, 1, 1) = -\pi^{4-2\varepsilon}\Gamma^2(1+\varepsilon) \, \mu_T^2 \sum_{j=0}^{\infty} \left(\frac{y}{\mu_T^2}\right)^j \left[\widetilde{C}_j^{(\text{h-h})} \left(\mu_T^2\right)^{-2\varepsilon} + \widetilde{C}_j^{(\text{p-p})} y^{-2\varepsilon}\right] \, . \tag{10}$$

In such way, it is more straightforward to use the threshold results for the lowest integrals obtained in [24]. Another advantage, as we shall see, is connected with the "book keeping" of ultraviolet singularities.

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Anyway, the transition between these two options (and, in particular, between the coefficients C_j and $\widetilde{C}_j$ in eqs. (5) and (10), respectively) is rather straightforward. For instance, any integral L_T (9) can be presented as $(\mu_T^2)^{n-\nu_1-\nu_2-\nu_3}$ times a function of dimensionless parameters ξ_i, eq. (2). When we consider the momentum integrals corresponding to the expansion (8), the only difference is that we need to replace μ_T^2 by k^2, namely:

$$\int \int \frac{\mathrm{d}^n p_1 \, \mathrm{d}^n p_2}{[p_1^2 + 2\xi_1(kp_1)]^{\nu_1} \, [p_2^2 + 2\xi_2(kp_2)]^{\nu_2} \, [(p_1 + p_2)^2 - 2\xi_3(k, p_1 + p_2)]^{\nu_3}}$$
$$= (k^2/\mu_T^2)^{n-\nu_1-\nu_2-\nu_3} \, L_T(n; \nu_1, \nu_2, \nu_3) \, . \tag{11}$$

If we consider the (h-h) contributions (8) corresponding to $L(n; 1, 1, 1)$, the first terms are proportional to

$$L_T(n; 1, 1, 1),$$
$$m_1^2 L_T(n; 2, 1, 1) + m_2^2 L_T(n; 1, 2, 1) + m_3^2 L(n; 1, 1, 2),$$
$$m_1^2 m_2^2 L_T(n; 2, 2, 1) + m_2^2 m_3^2 L_T(n; 1, 2, 2) + m_3^2 m_1^2 L_T(n; 2, 1, 2)$$
$$+ m_1^4 L_T(n; 3, 1, 1) + m_2^4 L_T(n; 1, 3, 1) + m_3^4 L_T(n; 1, 1, 3), \tag{12}$$

etc. It is easy to see that each new order will contain ultraviolet-divergent combinations

$$m_1^{2j} L_T(n; j + 1, 1, 1) + m_2^{2j} L_T(n; 1, j + 1, 1) + m_3^{2j} L_T(n; 1, 1, j + 1). \tag{13}$$

The situation is different if we use the variables y and μ_T^2, according to eq. (10). It is instructive to consider the formal Taylor expansion around the threshold $k^2 = \mu_T^2$ $(y = 0)$,

$$\sum_{j=0}^{\infty} \frac{(-y)^j}{j!} \left[\left(\frac{\partial}{\partial k^2} \right)^j L(n; \nu_1, \nu_2, \nu_3) \right]_{k^2 = (m_1 + m_2 + m_3)^2}, \tag{14}$$

which also corresponds to the (h-h) contribution. An algorithmically convenient way to construct expressions for the derivatives in (14) is to use the formulae

$$\left(\frac{\partial}{\partial k^2} \right)^j L(n; \nu_1, \nu_2, \nu_3) = \frac{(-1)^j}{\pi^{2j}} \, (\nu_1)_j (\nu_2)_j (\nu_3)_j \, L(n + 2j; \nu_1 + j, \nu_2 + j, \nu_3 + j) \tag{15}$$

and

$$\pi^{-2} k^2 (\nu_1 + j - 1)(\nu_2 + j - 1)(\nu_3 + j - 1) L(n + 2j; \nu_1 + j, \nu_2 + j, \nu_3 + j)$$
$$= (\nu_1 + \nu_2 + \nu_3 - n + j - 1) \, L(n + 2j - 2; \nu_1 + j - 1, \nu_2 + j - 1, \nu_3 + j - 1)$$
$$+ (\nu_1 + j - 1) m_1^2 \, L(n + 2j - 2; \nu_1 + j, \nu_2 + j - 1, \nu_3 + j - 1)$$
$$+ (\nu_2 + j - 1) m_2^2 \, L(n + 2j - 2; \nu_1 + j - 1, \nu_2 + j, \nu_3 + j - 1)$$
$$+ (\nu_3 + j - 1) m_3^2 \, L(n + 2j - 2; \nu_1 + j - 1, \nu_2 + j - 1, \nu_3 + j). \tag{16}$$

Applying eq. (16) j times, we reduce the space-time dimension back to $n = 4 - 2\varepsilon$. Both formulae (15) and (16) can be easily derived by using the modified Feynman parametric representation for the integral (1) given by eq. (4) of ref. [24] (cf. also in refs. [27]).

When we consider the Taylor expansion of $L(n; 1, 1, 1)$, the derivatives are given by

$$-k^2 \frac{\partial}{\partial k^2} L(n; 1, 1, 1) = (3{-}n)L(n; 1, 1, 1) + m_1^2 L(n; 2, 1, 1) + m_2^2 L(n; 1, 2, 1) + m_3^2 L(n; 1, 1, 2),$$
(17)

$$\tfrac{1}{2}(k^2)^2 \left(\frac{\partial}{\partial k^2}\right)^2 L(n; 1, 1, 1) = \tfrac{1}{2}(4 - n)(3 - n)L(n; 1, 1, 1)$$

$$+(4{-}n)\left[m_1^2 L(n; 2, 1, 1) + m_2^2 L(n; 1, 2, 1) + m_3^2 L(n; 1, 1, 2)\right]$$

$$+m_1^2 m_2^2 L(n; 2, 2, 1) + m_2^2 m_3^2 L(n; 1, 2, 2) + m_3^2 m_1^2 L(n; 2, 1, 2)$$

$$+m_1^4 L(n; 3, 1, 1) + m_2^4 L(n; 1, 3, 1) + m_3^4 L(n; 1, 1, 3), \quad (18)$$

etc. In these derivatives (see eq. (14)), we should put $k^2 = (m_1 + m_2 + m_3)^2$ ($L \to L_T$).

Note that now the ultraviolet singularities are present only in the y^0 and y^1 terms of the expansion of $L(n; 1, 1, 1)$. One can see that, apart from the lower integrals, the derivatives (17) and (18) contain the same combinations as (12). Furthermore, one can get the results (17) and (18) directly from eq. (12), by multiplying each contribution by the factor (11), $(k^2/\mu_T^2)^{n-3} = (1 - y/\mu_T^2)^{n-3}$, expanding it in y, and collecting terms at given powers of y.

The analytic results for $L_T(4 - 2\varepsilon; 1, 1, 1)$ and $L_T(4 - 2\varepsilon; 1, 1, 2)$, expanded in ε up to the finite part, are given in eqs. (23) and (22) of ref. [24], respectively. The results for $L_T(4 - 2\varepsilon; 2, 1, 1)$ and $L_T(4 - 2\varepsilon; 1, 2, 1)$ can be obtained from $L_T(4 - 2\varepsilon; 1, 1, 2)$ via permutation of the masses m_i. Those results involve the following functions:

$$T^-(z) = \text{Li}_2(-z) - \text{Li}_2(-1/z) + 2\ln z \ln(1 + z) - \ln^2 z$$

$$= 2\text{Li}_2(-z) + \tfrac{1}{6}\pi^2 + 2\ln z \ln(1 + z) - \tfrac{1}{2}\ln^2 z$$

$$= 2\text{Li}_2(1/(1 + z)) - \tfrac{1}{6}\pi^2 + \ln^2(1 + z) - \tfrac{1}{2}\ln^2 z, \quad (19)$$

$$\theta_i \equiv \arctan\left(m_i \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}}\right), \qquad \theta_1 + \theta_2 + \theta_3 = \pi. \quad (20)$$

Note that the arguments of T^- are just the mass ratios m_j/m_l, and the following inversion property is valid: $T^-(m_j/m_l) = -T^-(m_l/m_j)$. In particular, $T^-(1) = 0$ (remember that $\text{Li}_2(-1) = -\tfrac{1}{12}\pi^2$). Similar functions have also appeared in refs. [28].

In the next two sections, we shall consider evaluation of the integrals L_T with higher powers of propagators.

3. Recurrence relations at the threshold

In this section, we shall consider relations connecting the integrals (1) with different indices ν_i. When L is used without its arguments, the non-shifted n and ν_i are understood. The standard notation for the raising and lowering operators reads

$$\mathbf{1}^\pm L = L(n; \nu_1 \pm 1, \nu_2, \nu_3), \quad \mathbf{2}^\pm L = L(n; \nu_1, \nu_2 \pm 1, \nu_3), \quad \mathbf{3}^\pm L = L(n; \nu_1, \nu_2, \nu_3 \pm 1). \quad (21)$$

When the external momentum squared takes its threshold value, $k^2 = (m_1 + m_2 + m_3)^2$, we shall use the notation (9).

Using the identities [22]

$$\int \int \mathrm{d}^n p_1 \, \mathrm{d}^n p_2 \frac{\partial}{\partial p_{i_\mu}} \left\{ \frac{A_\mu^{(j)}}{[p_1^2 - m_1^2]^{\nu_1} [p_2^2 - m_2^2]^{\nu_2} [(k - p_1 - p_2)^2 - m_3^2]^{\nu_3}} \right\} = 0, \quad (22)$$

with $A_\mu^{(j)} = \left\{ p_{1_\mu}, p_{2_\mu}, k_\mu \right\}$, we obtain a set of integration-by-parts relations. For example, one of the equations can be presented as

$$\left[2m_1^2 \nu_1 \mathbf{1}^+ + (m_1^2 + m_2^2 + m_3^2 - k^2)\nu_2 \mathbf{2}^+ + 2m_3^2 \nu_3 \mathbf{3}^+ + \frac{2m_1^2 \nu_1 \nu_2}{\nu_3 - 1} \, \mathbf{3}^- \mathbf{1}^+ \mathbf{2}^+ \right] L$$

$$= \left[2n - 2\nu_1 - \nu_2 - 2\nu_3 - \nu_2 \mathbf{1}^- \mathbf{2}^+ + \frac{n - 2\nu_1 - \nu_3 + 1}{\nu_3 - 1} \nu_2 \mathbf{3}^- \mathbf{2}^+ \right] L, \quad (23)$$

and five other equations can be obtained from (23) by permutations of the indices (1,2,3).

Using three of these six equations, we can e.g. express $\mathbf{3}^- \mathbf{1}^+ \mathbf{2}^+ L$, $\mathbf{2}^- \mathbf{1}^+ \mathbf{3}^+ L$ and $\mathbf{1}^- \mathbf{2}^+ \mathbf{3}^+ L$ in terms of $\mathbf{1}^+ L$, $\mathbf{2}^+ L$ and $\mathbf{3}^+ L$. However, three remaining equations for $\mathbf{1}^+ L$, $\mathbf{2}^+ L$ and $\mathbf{3}^+ L$ happen to be linearly dependent, for arbitrary k^2. Useful corollaries of eq. (23) and its permutations are

$$\left[m_1^2 \nu_1 (\nu_1 + 1) \mathbf{1}^+ \mathbf{1}^+ - m_3^2 \nu_3 (\nu_3 + 1) \mathbf{3}^+ \mathbf{3}^+ \right] L = \tfrac{1}{2} \left[(n - 2\nu_1 - 2)\nu_1 \mathbf{1}^+ - (n - 2\nu_3 - 2)\nu_3 \mathbf{3}^+ \right] L, \quad (24)$$

$$\left[m_2^2 \nu_2 (\nu_2 + 1) \mathbf{2}^+ \mathbf{2}^+ - m_3^2 \nu_3 (\nu_3 + 1) \mathbf{3}^+ \mathbf{3}^+ \right] L = \tfrac{1}{2} \left[(n - 2\nu_2 - 2)\nu_2 \mathbf{2}^+ - (n - 2\nu_3 - 2)\nu_3 \mathbf{3}^+ \right] L. \quad (25)$$

Below we shall use these two equations as a part of the main algorithm.

Let us denote $\sigma \equiv \nu_1 + \nu_2 + \nu_3$. Since the threshold values of the integrals with $\sigma = 3$ ($\nu_1 = \nu_2 = \nu_3 = 1$) and $\sigma = 4$ ($\nu_1 = \nu_2 = 1, \nu_3 = 2$ and permutations) are known [24], the first non-trivial step is to get results for six integrals with $\sigma = 5$ ($L(n; 2, 2, 1), L(n; 1, 1, 3)$ and permutations). Introduce the following notation for the symmetric sums of these integrals:

$$S_{221} \equiv L(n; 1, 2, 2) + L(n; 2, 1, 2) + L(n; 2, 2, 1), \quad (26)$$

$$S_{113} \equiv m_1^2 L(n; 3, 1, 1) + m_2^2 L(n; 1, 3, 1) + m_3^2 L(n; 1, 1, 3). \quad (27)$$

Then, using eqs. (24)–(25) we get

$$L(n; 1, 1, 3) = \frac{1}{3m_3^2} \left\{ S_{113} + \tfrac{1}{4}(n - 4) \left[2L(n; 1, 1, 2) - L(n; 2, 1, 1) - L(n; 1, 2, 1) \right] \right\} \quad (28)$$

and analogous results for $L(n; 3, 1, 1)$ and $L(n; 1, 3, 1)$. Furthermore, using eq. (23) and its permutations, we get

$$L(n; 2, 2, 1) = \frac{1}{k^2 - m_1^2 - m_2^2 + m_3^2} \Big\{ 2m_3^2 S_{221} + \tfrac{4}{3} S_{113} - \tfrac{1}{3}(n-4)L(n; 1, 1, 2)$$

$$- \tfrac{1}{3}(4n - 13)\left[L(n; 2, 1, 1) + L(n; 1, 2, 1)\right] + L(n; 2, 2, 0) \Big\} \qquad (29)$$

and analogous results for $L(n; 1, 2, 2)$ and $L(n; 2, 1, 2)$. Note that at the threshold

$$\left. \left(k^2 - m_1^2 - m_2^2 + m_3^2 \right) \right|_{k^2 = (m_1 + m_2 + m_3)^2} = 2(m_1 + m_3)(m_2 + m_3).$$

Furthermore, taking the sum of eq. (29) and its permutations, we can express S_{221} via S_{113} and lower integrals. In particular, at the threshold we get

$$m_1 m_2 m_3 S_{221}^{(T)} - \tfrac{2}{3}\mu_T S_{113}^{(T)} = -\tfrac{1}{12}(5n - 17)\mu_T \left[L_T(n; 2, 1, 1) + L_T(n; 1, 2, 1) + L_T(n; 1, 1, 2)\right]$$

$$- \tfrac{1}{4}(n-3) \left[m_1 L_T(n; 2, 1, 1) + m_2 L_T(n; 1, 2, 1) + m_3 L_T(n; 1, 1, 2)\right]$$

$$+ \tfrac{1}{4} \left[(m_2 + m_3)L(n; 0, 2, 2) + (m_1 + m_3)L(n; 2, 0, 2) + (m_1 + m_2)L(n; 2, 2, 0)\right], \quad (30)$$

where the superscript "(T)" means that the corresponding sums, $S_{221}^{(T)}$ and $S_{113}^{(T)}$, are considered at the threshold. Therefore, using the integration-by-parts relations we reduce all the six integrals with $\sigma = 5$ to a single unknown function (say, S_{221}).

A missing "block" completing the algorithm can be obtained using some equations presented in ref. [23]. Consider eqs. (74) and (81) of [23]. Their l.h.s.'s contain a factor

$$D_{123} \equiv \left[k^2 - (m_1 + m_2 + m_3)^2\right] \left[k^2 - (-m_1 + m_2 + m_3)^2\right]$$

$$\times \left[k^2 - (m_1 - m_2 + m_3)^2\right] \left[k^2 - (m_1 + m_2 - m_3)^2\right]. \qquad (31)$$

In the case of interest (i.e. at the threshold) these l.h.s.'s vanish, since $D_{123} = 0$. Therefore, at the threshold these equations cannot be used in the way as it is suggested in ref. [23], since this is a degenerate case. Nevertheless, the fact that the r.h.s.'s should also be equal to zero, provides some non-trivial conditions on the integrals involved. It is interesting that at the threshold both eqs. (74) and (81) of [23] lead to the same condition:

$$\Big\{ \nu_1 m_1 \left[(2n - \nu_1 - 2\nu_2 - 2\nu_3 - 1)m_1 + (n - \nu_2 - 2\nu_3)m_2 + (n - 2\nu_2 - \nu_3)m_3\right] \mathbf{1}^+$$

$$+ \nu_2 m_2 \left[(n - \nu_1 - 2\nu_3)m_1 + (2n - 2\nu_1 - \nu_2 - 2\nu_3 - 1)m_2 + (n - 2\nu_1 - \nu_3)m_3\right] \mathbf{2}^+$$

$$+ \nu_3 m_3 \left[(n - \nu_1 - 2\nu_2)m_1 + (n - 2\nu_1 - \nu_2)m_2 + (2n - 2\nu_1 - 2\nu_2 - \nu_3 - 1)m_3\right] \mathbf{3}^+$$

$$- \nu_2 \nu_3 m_2 m_3 \mathbf{1}^- \mathbf{2}^+ \mathbf{3}^+ - \nu_1 \nu_3 m_1 m_3 \mathbf{2}^- \mathbf{1}^+ \mathbf{3}^+ - \nu_1 \nu_2 m_1 m_2 \mathbf{3}^- \mathbf{1}^+ \mathbf{2}^+$$

$$- \tfrac{1}{2}(n - \nu_1 - \nu_2 - \nu_3)(3n - 2\nu_1 - 2\nu_2 - 2\nu_3 - 2) \Big\} L_T(n; \nu_1, \nu_2, \nu_3) = 0. \quad (32)$$

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For example, when $\nu_1 = \nu_2 = \nu_3 = 1$, eq. (32) yields

$$(n-3)\left\{m_1(2m_1 + m_2 + m_3)L_T(n;2,1,1) + m_2(m_1 + 2m_2 + m_3)L_T(n;1,2,1)\right.$$

$$\left. +m_3(m_1 + m_2 + 2m_3)L_T(n;1,1,2) - \tfrac{1}{2}(3n-8)L_T(n;1,1,1)\right\}$$

$$= m_2 m_3 L(n;0,2,2) + m_1 m_3 L(n;2,0,2) + m_1 m_2 L(n;2,2,0), \quad (33)$$

where on the r.h.s. we have just the products of massive tadpoles,

$$L(n;\nu_1,\nu_2,0) = \mathrm{i}^{2-2\nu_1-2\nu_2}\pi^n \, (m_1^2)^{n/2-\nu_1} \, (m_2^2)^{n/2-\nu_2} \, \frac{\Gamma(\nu_1 - n/2)\Gamma(\nu_2 - n/2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \quad (34)$$

and permutations. We have checked that the results for $L_T(n;1,1,1)$, $L_T(n;1,1,2)$ (and permutations) presented in eqs. (22) and (21) of [24] satisfy eq. (33).

Furthermore, considering eq. (32) for $\nu_1 = \nu_2 = 1$, $\nu_3 = 2$ (and permutations), we obtain an extra (independent of (30)) condition on the sums $S_{221}^{(T)}$ and $S_{113}^{(T)}$. In this way we can obtain results for the integrals $L_T(n;2,2,1)$, $L_T(n;1,1,3)$ (and permutations) in terms of lower integrals. However, at this step (solving the system of linear equations) we get the factor $(n-4)$ in the denominator. This means that we need to know the ε part of the lower integrals, in order to get the finite part of $L_T(n;2,2,1)$, etc. Alternatively, we can calculate the finite part of $L_T(n;2,2,1)$ via straightforward calculation. As we shall see, this will be enough to calculate the higher integrals (see in Appendix A for details).

4. Calculation of $L_T(4-2\varepsilon;2,2,1)$ with different masses

We start from the two-fold integral representation (5) of ref. [24], which in the case $\nu_1 = \nu_2 = 2$, $\nu_3 = 1$ reads

$$L_T(4-2\varepsilon;2,2,1) = \pi^{4-2\varepsilon}\,\Gamma(1+2\varepsilon)\,(m_1 m_2)^{-\varepsilon}\,m_3^{1-\varepsilon}$$

$$\times \int_0^\infty\!\!\int_0^\infty \frac{\mathrm{d}\xi\,\mathrm{d}\eta\,\xi^\varepsilon\,\eta^\varepsilon}{(m_1\xi + m_2\eta + m_3)^{1-3\varepsilon}\,[m_2 m_3\xi(1-\eta)^2 + m_1 m_3\eta(1-\xi)^2 + m_1 m_2(\xi-\eta)^2]^{1+2\varepsilon}}. \quad (35)$$

The threshold singularity originates from the region $\xi \sim \eta \sim 1$. Indeed, analyzing eq. (5) of [24] we see that the threshold singularity takes place (in the four-dimensional space) when $\nu_1 + \nu_2 + \nu_3 \geq 5$. It is of an infrared origin and (as we shall see) appears in dimensionally-regularized integrals as a pole in $\varepsilon = (4-n)/2$.

The double integral in the second line of eq. (35) can be decomposed as

$$\frac{1}{\mu_T^{1-3\varepsilon}}\int_0^\infty\!\!\int_0^\infty \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{[m_2 m_3\xi(1-\eta)^2 + m_1 m_3\eta(1-\xi)^2 + m_1 m_2(\xi-\eta)^2]^{1+2\varepsilon}}$$

$$+\frac{1}{\mu_T}\int_0^\infty\!\!\int_0^\infty \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{[m_2 m_3\xi(1-\eta)^2 + m_1 m_3\eta(1-\xi)^2 + m_1 m_2(\xi-\eta)^2]} \frac{[m_1(1-\xi) + m_2(1-\eta)]}{(m_1\xi + m_2\eta + m_3)}$$

$$+\mathcal{O}(\varepsilon), \quad (36)$$

where the threshold singularity is only in the first term (which is simpler than the original integral (35)), whereas the second term is finite as $n \to 4$.

To evaluate the integrals in (36), we can proceed in the same way as in [24]. However, the following substitution of variables[4] happens to be more efficient than eq. (13) in [24]:

$$\xi = 1 + (m_2/m_3)\lambda z, \qquad \eta = 1 + (m_1/m_3)\lambda(1-z) . \tag{37}$$

Then, the region of integration in variables z, λ is defined by

$$-m_3/(m_2\lambda) \leq z \leq 1 + m_3/(m_1\lambda) \qquad (\text{for} \quad \lambda > 0) ,$$

$$1 + m_3/(m_1\lambda) \leq z \leq -m_3/(m_2\lambda) \qquad (\text{for} \quad \lambda_{\min} < \lambda < 0) ,$$

with $\lambda_{\min} = -m_3(m_1+m_2)/(m_1m_2)$. Note that the resulting integrand is invariant under $(z, m_1) \leftrightarrow (1-z, m_2)$. The threshold singularity corresponds to the region $\lambda \sim 0$: the transformed integrand of the first integral in (36) contains $\lambda^{-1-4\varepsilon}$.

Evaluating the integrals in eq. (36) we arrive at the following result:

$$L_T(4 - 2\varepsilon; 2, 2, 1) = \pi^{4-2\varepsilon} \, \Gamma^2(1 + \varepsilon) \, (m_1 m_2 m_3)^{-3\varepsilon} \, \mu_T^{-2+5\varepsilon}$$

$$\times \left\{ -m_3\sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}} \left[\frac{\pi}{2\varepsilon} + \pi\mathcal{L}(m_1, m_2, m_3) + \mathcal{C}(\theta_1, \theta_2, \theta_3) \right] \right.$$

$$\left. + T^- \left(\frac{m_1}{m_3} \right) + T^- \left(\frac{m_2}{m_3} \right) + \frac{4\pi^2}{3} - 4\pi\theta_3 + \ln^2 \frac{m_1}{m_2} - \tfrac{1}{2} \ln^2 \frac{m_1}{m_3} - \tfrac{1}{2} \ln^2 \frac{m_2}{m_3} \right] \right\} + \mathcal{O}(\varepsilon), \tag{38}$$

with

$$\mathcal{C}(\theta_1, \theta_2, \theta_3) \equiv \text{Cl}_2(2\theta_1) + \text{Cl}_2(2\theta_2) + \text{Cl}_2(2\theta_3) + \text{Cl}_2(\pi - 2\theta_1) + \text{Cl}_2(\pi - 2\theta_2) + \text{Cl}_2(\pi - 2\theta_3) \tag{39}$$

and

$$\mathcal{L}(m_1, m_2, m_3) \equiv \ln\left(\frac{(m_1 + m_2)(m_2 + m_3)(m_3 + m_1)}{(m_1 + m_2 + m_3)^3}\right) - 6\ln 2. \tag{40}$$

The functions T^- and θ_i are defined in eqs. (19) and (20), respectively (cf. eqs. (14) and (20) of [24]). The Clausen function is defined as $\text{Cl}_2(\theta) = \text{Im}\left[\text{Li}_2\left(e^{i\theta}\right)\right]$ (for details, see e.g. in [29]). Note that the combination of Clausen functions (39) is related to the volumes of asymptotic tetrahedra in hyperbolic space of constant curvature (see in [30]). Worth noting is that at the threshold the result for the symmetric combination (26) is

$$S_{221}^{(T)} = -\frac{\pi^{4-2\varepsilon} \, \Gamma^2(1 + \varepsilon)}{(m_1 m_2 m_3)^{1/2+3\varepsilon} \mu_T^{1/2-5\varepsilon}} \left[\frac{\pi}{2\varepsilon} + \pi\mathcal{L}(m_1, m_2, m_3) + \mathcal{C}(\theta_1, \theta_2, \theta_3) \right] + \mathcal{O}(\varepsilon). \tag{41}$$

Once more, we would like to emphasize that the $1/\varepsilon$ poles in eqs. (38) and (41) (accompanied by an extra factor of π) correspond to the threshold singularity: there are no ultraviolet divergences in these results.

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[4]A. D. is grateful to J.B. Tausk for discussion of this substitution in connection with ref. [24].

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Using recurrence relations, we can obtain results for $L_T(4 - 2\varepsilon; 1, 1, 3)$ with different masses, as well as for the higher integrals. Some relevant results are collected in Appendix B.

In the equal-mass case, $\theta_1 = \theta_2 = \theta_3 = \frac{1}{3}\pi$ (cf. eq. (20)), $T^-(1) = 0$ (cf. eq. (19)), $\mathcal{C}(\pi/3, \pi/3, \pi/3) = 5\text{Cl}_2(\pi/3)$ (we take into account that $\text{Cl}_2(2\pi/3) = \frac{2}{3}\text{Cl}_2(\pi/3)$). Therefore we get

$$L_T(n; 2, 2, 1)|_{m_1=m_2=m_3\equiv m} \equiv L_T^{(\text{eq})}(n; 2, 2, 1)$$

$$= -\pi^{4-2\varepsilon}\,\Gamma^2(1+\varepsilon)\,m^{-2-4\varepsilon}\left\{\frac{\pi}{6\sqrt{3}}\left(\frac{1}{\varepsilon} - \ln 3 - 6\ln 2\right) + \frac{5}{3\sqrt{3}}\text{Cl}_2\left(\frac{\pi}{3}\right)\right\} + \mathcal{O}(\varepsilon). \quad (42)$$

It is interesting that $\text{Cl}_2(\pi/3)$ (i.e. the maximal value of $\text{Cl}_2(\theta)$) also appears in the two-loop integrals containing a threshold singularity[5].

In fact, knowing the result (42) we can obtain ε-part of the integrals $L_T^{(\text{eq})}(n; 1, 1, 2)$ and $L_T^{(\text{eq})}(n; 1, 1, 1)$. Using recurrence relations, we get

$$L_T^{(\text{eq})}(n; 1, 1, 2) = \frac{1}{(n-3)(3n-10)}\left\{-8m^2(n-4)L_T^{(\text{eq})}(n; 2, 2, 1) + (2n-7)L^{(\text{eq})}(2, 2, 0)\right\}. \quad (43)$$

It is important that the integral (42) enters with a factor $(n-4)$. Expanding in ε we get

$$L_T^{(\text{eq})}(4 - 2\varepsilon; 1, 1, 2) = \pi^{4-2\varepsilon}\,\Gamma^2(1+\varepsilon)\,m^{-4\varepsilon}\left\{-\frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon} + \frac{1}{2} + \frac{11}{2}\varepsilon\right.$$

$$\left. -\frac{4\pi}{3\sqrt{3}}[1 + 5\varepsilon - \varepsilon\ln 3 - 6\varepsilon\ln 2] - \frac{40\varepsilon}{3\sqrt{3}}\text{Cl}_2\left(\frac{\pi}{3}\right)\right\} + \mathcal{O}(\varepsilon^2). \quad (44)$$

Furthermore, using eq. (33) we obtain

$$L_T^{(\text{eq})}(n; 1, 1, 1) = \frac{6m^2}{(n-3)(3n-8)}\left\{4(n-3)\,L_T^{(\text{eq})}(n; 1, 1, 2) - L^{(\text{eq})}(n; 2, 2, 0)\right\}. \quad (45)$$

Finally, we arrive at

$$L_T^{(\text{eq})}(4 - 2\varepsilon; 1, 1, 1) = \pi^{4-2\varepsilon}\,\Gamma^2(1+\varepsilon)\,m^{2-4\varepsilon}\left\{-\frac{3}{2\varepsilon^2} - \frac{9}{4\varepsilon} + \frac{45}{8} + \frac{855}{16}\varepsilon\right.$$

$$\left. +\frac{4\pi}{\sqrt{3}}[-2 - 13\varepsilon + 2\varepsilon\ln 3 + 12\varepsilon\ln 2] - \frac{80\varepsilon}{\sqrt{3}}\text{Cl}_2\left(\frac{\pi}{3}\right)\right\} + \mathcal{O}(\varepsilon^2). \quad (46)$$

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[5]For some other examples illustrating occurence of the Clausen function in two-loop calculations, including vacuum integrals and the pseudothreshold values, see e.g. in [31, 32]. Note that $\text{Cl}_2(\pi/3) = \left[\psi'(\frac{1}{3}) - \frac{2}{3}\pi^2\right]/(2\sqrt{3})$ (cf. in [32, 33]).

5. Evaluation of the (p-p) contribution

According to the prescription of [1], we should start by expanding the propagators in the integrand of eq. (4) in p_{i0}^2,

$$\frac{1}{[p_{i0}^2 - \vec{p}_i^2 + 2\xi_i(kp_i) - \xi_i^2 y]^{\nu_i}} \Rightarrow \sum_{j_i=0}^{\infty} \frac{(\nu_i)_{j_i}}{j_i!} \frac{p_{i0}^{2j_i}}{[-\vec{p}_i^2 + 2\xi_i k_0 p_{i0} - \xi_i^2 y]^{\nu_i+j_i}} \;, \tag{47}$$

where, as before, we imply that $p_3 = -p_1 - p_2$. However, it happens inconvenient to expand integrand in the time components of the loop momenta (according to eq. (47)) and then take p_{10} and p_{20} integrals, because one arrives at rather cumbersome integrals in $\vec{p}_1$ and $\vec{p}_2$. We shall proceed in a slightly different way.

Let us instead exponentiate, by use of the alpha parameters, every propagator of the unexpanded Feynman integral (4) and perform integration in $\vec{p}_1$ and $\vec{p}_2$. In particular, for the integral $L(4 - 2\varepsilon; 1, 1, 1)$ we have

$$- \mathrm{i}^{2\varepsilon} \pi^{n-1} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{\mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3}{(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)^{3/2-\varepsilon}} \exp\left[-\mathrm{i}y(\xi_1^2\alpha_1 + \xi_2^2\alpha_2 + \xi_3^2\alpha_3)\right]$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_{10}\mathrm{d}p_{20} \exp\left\{\mathrm{i}[(\alpha_1 + \alpha_3)p_{10}^2 + 2\alpha_3 p_{10}p_{20} + (\alpha_2 + \alpha_3)p_{20}^2]\right\}$$

$$\times \exp\left\{\mathrm{i}\left[2k_0 p_{10}(\xi_1\alpha_1 - \xi_3\alpha_3) + 2k_0 p_{20}(\xi_2\alpha_2 - \xi_3\alpha_3)\right]\right\} \;. \tag{48}$$

Only now we perform Taylor expansion: namely, in the exponent with p_{10}^2, p_{20}^2 and $p_{10}p_{20}$. Then the evaluation of the integral of an arbitrary resulting term is simple: we perform integrations in p_{10} and p_{20} and obtain terms proportional to derivatives of two δ-functions, $\delta(2k_0(\xi_1\alpha_1 - \xi_3\alpha_3))$ and $\delta(2k_0(\xi_2\alpha_2 - \xi_3\alpha_3))$. Then, we take the integrals in α_1 and α_2, and finally in α_3. As a result we have obtained an explicit formula for the (p-p) contribution in arbitrary order, written through a finite eight-fold sum.

In eq. (5) the (p-p) contribution starts from the order y^2, so that $C_0^{(\mathrm{p-p})} = C_1^{(\mathrm{p-p})} = 0$. For the first two non-trivial orders we have

$$C_2^{(\mathrm{p-p})} = \frac{\pi\,(\xi_1\xi_2\xi_3)^{1/2-\varepsilon}}{4\varepsilon(1 - \varepsilon)(1 - 2\varepsilon)} \;, \tag{49}$$

$$C_3^{(\mathrm{p-p})} = \frac{\pi\,(\xi_1\xi_2\xi_3)^{-1/2-\varepsilon}}{64\varepsilon(1 - \varepsilon)(1 - 2\varepsilon)} \left[(1 - 2\varepsilon)\,(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) - (5 - 18\varepsilon)\xi_1\xi_2\xi_3\right]. \tag{50}$$

In the equal-mass case, we get

$$C_j^{(\mathrm{p-p}),(\mathrm{eq})} = (-1)^{j+1}\frac{\pi}{\sqrt{3}}\left[a_j\left(\frac{1}{\varepsilon} + 3\ln 3\right) + b_j\right] + \mathcal{O}(\varepsilon) \;. \tag{51}$$

The coefficients a_j and b_j ($j \le 12$) are given in Table 1. Note that the threshold expansion of the imaginary part of the sunset diagram is completely characterized by the pole part

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Table 1: The coefficients in eqs. (51), (55) and (56).

j	a_j	b_j	$\widetilde{a}_j = \widetilde{a}'_j$	$\widetilde{b}_j$	$\widetilde{b}'_j$	$\widetilde{c}_j$
2	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{2}{9}$	$\frac{1}{4}$	$-\frac{1}{12}$
3	$\frac{1}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{41}{144}$	$\frac{3}{16}$	$\frac{1}{8}$
4	$\frac{7}{768}$	$\frac{15}{512}$	$\frac{13}{256}$	$\frac{1279}{4608}$	$\frac{79}{512}$	$\frac{11}{64}$
5	$\frac{1}{192}$	$\frac{9}{512}$	$\frac{11}{256}$	$\frac{2407}{9216}$	$\frac{17}{128}$	$\frac{59}{320}$
6	$\frac{83}{24576}$	$\frac{583}{49152}$	$\frac{305}{8192}$	$\frac{72019}{294912}$	$\frac{5767}{49152}$	$\frac{1901}{10240}$
7	$\frac{233}{98304}$	$\frac{1693}{196608}$	$\frac{1073}{32768}$	$\frac{269359}{1179648}$	$\frac{20719}{196608}$	$\frac{261607}{1433600}$
8	$\frac{1381}{786432}$	$\frac{6889}{1048576}$	$\frac{7623}{262144}$	$\frac{1345217}{6291456}$	$\frac{100361}{1048576}$	$\frac{2034657}{11468800}$
9	$\frac{2129}{1572864}$	$\frac{32695}{6291456}$	$\frac{13623}{524288}$	$\frac{52988741}{264241152}$	$\frac{183713}{2097152}$	$\frac{27534999}{160563200}$
10	$\frac{108257}{100663296}$	$\frac{1701199}{402653184}$	$\frac{781899}{33554432}$	$\frac{3184520135}{16911433728}$	$\frac{10819013}{134217728}$	$\frac{1697828907}{10276044800}$
11	$\frac{352373}{402653184}$	$\frac{5653147}{1610612736}$	$\frac{2809445}{134217728}$	$\frac{35912515043}{202937204736}$	$\frac{119896193}{1610612736}$	$\frac{14363946379}{90429194240}$
12	$\frac{4677635}{6442450944}$	$\frac{382328867}{128849018880}$	$\frac{40375137}{2147483648}$	$\frac{900245481797}{5411658792960}$	$\frac{2959291009}{42949672960}$	$\frac{220597949391}{1446867107840}$

of the (p-p) contribution, i.e. by the coefficients a_j in Table 1. We have compared our results for the imaginary part with analytical expressions (in terms of elliptic integrals) presented in [18] (cf. also in [34]) and found complete agreement.

Note that for an arbitrary, L-loop "water-melon" diagram (consisting of two vertices and $(L+1)$ massive lines between them) the (p-p-...-p) contribution can also be analytically calculated in every order. In particular, in the leading order we have the following result:

$$i^L \pi^{(n+1)L/2} \Gamma((\varepsilon - 3/2)L + 1) \left(\prod_{j=1}^{L+1} \xi_j^{1/2-\varepsilon} \right) \frac{1}{(k^2)^{L/2} y^{(\varepsilon-3/2)L+1}} , \qquad (52)$$

where, by analogy with eq. (2), $\xi_j = m_j/(\sum_{i=1}^{L+1} m_i)$.

6. Results and numerical comparison

Collecting parts of the algorithm described in the previous sections, we obtain terms of the threshold expansion of the sunset diagram (1). To see how efficiently the threshold expansion works, we have compared our results with a semi-numerical program based

on the algorithm of [19][6]. The ultraviolet-divergent terms were compared analytically, whereas the finite (in ε) part was treated numerically, using an integral representation from [19].

Our approximations correspond to the threshold expansion given by eqs. (5) and (10). We refer to the sum of terms up to order y^N as to an N-th approximation. We illustrate the convergence in the plots, Fig. 2 and Fig. 3, where we present the subtracted (i.e., without poles in ε) real part and the imaginary part of $(\mu_T^2)^{-1+2\varepsilon} L_T(4-2\varepsilon; 1, 1, 1)$. Note that the imaginary part is finite as $\varepsilon \to 0$. These quantities are shown as functions of k^2/μ_T^2 (which equals $k^2/(9m^2)$ in the equal-mass case). In all plots, the highest-order curve is denoted by the solid line, whereas the lower approximations are drawn by various dashed lines (see the plots). The results of the above-mentioned numerical program [19] are displayed as crosses.

As the first example for numerical comparison, we have chosen the case $\xi_1 = 0.1$, $\xi_2 = 0.3$, $\xi_3 = 0.6$, when the masses are essentially different (but still, one cannot neglect either of them). Here, the expansion (10) in terms of y/μ_T^2 is used. The results are shown in Fig. 2. On one hand, we can see that the expansion of the real part converges reasonably well. On the other hand, the expansion of the imaginary part breaks down just above $k^2/\mu_T^2 \simeq 1.5$. In fact, the same happens to the real part, when $k^2/\mu_T^2 \simeq 2.2$. This is because the expansion parameter y/μ_T^2 is already not so small.

If we use, for $k^2 > \mu_T^2$, the expansion in terms of y/k^2 the convergence is much better. For instance, at $k^2 = 2\mu_T^2$ the $N = 7$ approximation provides a four-digit precision for the imaginary part. In general, it appears that an optimal choice is to use the expansion in y/μ_T^2 (cf. eq. (10)) for $k^2 < \mu_T^2$, and the expansion in y/k^2 (cf. eq. (5)) for $k^2 > \mu_T^2$. For the imaginary part, it is enough to use only the y/k^2 expansion, since it vanishes for $k^2 < \mu_T^2$.

As the second example, let us consider eq. (10) in the case of equal masses, $m_1 = m_2 = m_3 \equiv m$ ($\xi_i = 1/3$), remembering that $\widetilde{C}_0^{(p-p)} = \widetilde{C}_1^{(p-p)} = 0$. The two lowest (h-h) contributions are

$$\widetilde{C}_0^{(h-h),(eq)} = 3^{4\varepsilon} \left[\frac{1}{6\varepsilon^2} + \frac{1}{4\varepsilon} - \frac{5}{8} + \frac{8\pi}{9\sqrt{3}} \right] + \mathcal{O}(\varepsilon), \tag{53}$$

$$\widetilde{C}_1^{(h-h),(eq)} = 3^{4\varepsilon} \left[\frac{1}{4\varepsilon} + \frac{23}{24} - \frac{4\pi}{9\sqrt{3}} \right] + \mathcal{O}(\varepsilon). \tag{54}$$

It is easy to check that these two terms absorb all ultraviolet singularities of the original sunset integral. For $j \geq 2$, the coefficients have the following form:

$$\widetilde{C}_j^{(h-h),(eq)} = \frac{\pi}{\sqrt{3}} \left[\widetilde{a}_j \left(\frac{1}{\varepsilon} + 3\ln 3 - 6\ln 2 \right) + \widetilde{b}_j \right] + \frac{10\widetilde{a}_j}{\sqrt{3}} \mathrm{Cl}_2 \left(\frac{\pi}{3} \right) - \widetilde{c}_j + \mathcal{O}(\varepsilon), \tag{55}$$

$$\widetilde{C}_j^{(p-p),(eq)} = -\frac{\pi}{\sqrt{3}} \left[\widetilde{a}'_j \left(\frac{1}{\varepsilon} + 3\ln 3 \right) + \widetilde{b}'_j \right] + \mathcal{O}(\varepsilon). \tag{56}$$

The $1/\varepsilon$ poles in the (h-h) contribution, eq. (55), correspond to the threshold singularity. However, since there is no threshold singularity in the case $\nu_1 = \nu_2 = \nu_3 = 1$, these poles should cancel with those from the (p-p) contribution, eq. (56). This is the case if $\tilde{a}'_j = \tilde{a}_j$. We have checked that this property is valid for all available contributions. The corresponding coefficients (for $j \leq 12$) are collected in Table 1.

As a result, using eq. (10) we get the following threshold expansion in the equal-mass case:

$$L^{(\mathrm{eq})}(4 - 2\varepsilon; 1, 1, 1) = -\pi^{4-2\varepsilon}\Gamma^2(1+\varepsilon)(9m^2)^{1-2\varepsilon}\Big\{\widetilde{C}_0^{(\mathrm{h-h}),(\mathrm{eq})} + \frac{y}{9m^2}\widetilde{C}_1^{(\mathrm{h-h}),(\mathrm{eq})}$$

$$+\frac{2}{\sqrt{3}}\Big[\pi\ln\frac{y}{9m^2} - 3\pi\ln 2 + 5\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)\Big]A\Big(\frac{y}{9m^2}\Big) + \frac{\pi}{\sqrt{3}}B\Big(\frac{y}{9m^2}\Big) - C\Big(\frac{y}{9m^2}\Big)\Big\} + \mathcal{O}(\varepsilon), (57)$$

with the "form factors"

$$A(z) \equiv \sum_{j=2}^{\infty}\tilde{a}_j z^j, \qquad B(z) \equiv \sum_{j=2}^{\infty}(\tilde{b}_j - \tilde{b}'_j)z^j, \qquad C(z) \equiv \sum_{j=2}^{\infty}\tilde{c}_j z^j. \tag{58}$$

Note that the imaginary part of the expression in braces in (57) equals

$$\frac{2\pi^2}{\sqrt{3}}A\Big(\frac{y}{9m^2}\Big)\theta(k^2 - 9m^2), \tag{59}$$

where the function $A(z)$ can be expressed in terms of elliptic integrals [18].

We have also compared this equal-mass example with the program based on the algorithm of [19]. The results of this numerical comparison are shown in Fig. 3. We have used the "combined" expansion: below the threshold (for $k^2 < \mu_T^2$) our approximations correspond to the expansion (57)–(58), whereas beyond the threshold (for $k^2 > \mu_T^2$) we switch to the expansion in y/k^2. Technically, this transformation can be done just by substituting the arguments of the functions (58) via $y/(\mu_T^2) \to (y/k^2)/(1 + y/k^2)$ and re-expanding in y/k^2, up to the given order N.

From the plots it is clear that subsequent approximations are getting better in a wide range of the values of the external momentum squared, i.e. the threshold expansion indeed works well. Close to the threshold, it is enough to take just a few terms of the expansion for achieving precision which is better than that of the numerical program [19]. For $k^2 = 0.5\mu_T^2$, the $N = 7$ and $N = 12$ approximations to the real part reproduce four and six digits, respectively. When $k^2 = 2\mu_T^2$, the $N = 6$ approximation gives us three digits, whereas the $N = 12$ curve provides six-digit precision.

Note that the threshold expansion (at least, in its present form) does not seem to work in Euclidean region, i.e. when $k^2 < 0$. Then, the limits when some of the particles become massless are also non-regular, since one should take into account ultrasoft regions (see in Section 1). However, for these cases (when some masses vanish) exact results are available in refs. [11, 12].

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7. Conclusions

Let us briefly summarize the main results of this paper.

We have considered the application of prescriptions of ref. [1] to the construction of the threshold expansion of the sunset diagram (Fig. 1) at the three-particle threshold. We have treated the most complicated case, when all particles involved in the threshold cut are massive. The expansion involves two contributions: the (h-h) and the (p-p) ones.

The main technical difficulties are related to the (h-h) contribution. Basically, it corresponds to a formal Taylor expansion around the threshold. To construct the coefficients of this expansion, one needs to calculate threshold values of the corresponding sunset integrals (1) with higher powers of the propagators. Although results for the lowest integrals were known up to the finite parts in ε [24], the recursive procedure (based on the techniques [22, 23]) required either ε-part of those integrals or an explicit result for one of the higher integrals (e.g. $L_T(4 - 2\varepsilon; 2, 2, 1)$). We have calculated this integral analytically, in terms of dilogarithms and Clausen functions (38). Note that it contains a $1/\varepsilon$ pole which is associated with the threshold singularity (treated in the framework of dimensional regularization). The result (38) has completed the algorithm for calculating higher terms of the (h-h) contribution.

Collecting the (h-h) and (p-p) contributions, we get the terms of the threshold expansion. A comparison with the results of semi-numerical program [19] has been performed. It was shown that the combined expansion (using the variables y/μ_T^2 and y/k^2 below and above the threshold, respectively) provides good analytical approximations in a wide region of values of the external momentum squared. This illustrates how efficiently the general procedure of threshold expansion [1] works for diagrams with three-particle thresholds in the totally massive case.

Appendix A: Recurrence relations for different masses

The main parameter of recursion is the sum of the indices, $\sigma \equiv \nu_1 + \nu_2 + \nu_3$. Under "lower integrals" we understand the integrals with lower values of σ. When one of the ν's is zero, the corresponding integral is trivial (see eq. (34)). The integrals with $\sigma = 3$ ($\nu_1 = \nu_2 = \nu_3 = 1$) and $\sigma = 4$ ($\nu_1 = \nu_2 = 1$, $\nu_3 = 2$ and permutations) are known [24]. We are therefore interested in calculation of the integrals with $\sigma \geq 5$.

Using eqs. (24), (25) we can express any integral $L_T(n; \nu_1, \nu_2, \nu_3)$ with positive ν's in terms of $L_T(n; 1, 1, \sigma - 2)$, $L_T(n; 2, 1, \sigma - 3)$, $L_T(n; 1, 2, \sigma - 3)$ and $L_T(n; 2, 2, \sigma - 4)$.

Furthermore, using the integration-by-parts identities we can express $L_T(n; 2, 1, \sigma - 3)$ and $L_T(n; 1, 2, \sigma - 3)$ in terms of $L_T(n; 1, 1, \sigma - 2)$, $L_T(n; 2, 2, \sigma - 4)$ and lower integrals, via

$$2(\nu_3 - 1)m_3(m_1 + m_2)(2m_1m_2 + m_1m_3 + m_2m_3)L_T(n; 2, 1, \nu_3)$$

$$= 2m_2^2(m_1 + m_2)(m_1 + m_3)L_T(n; 2, 2, \nu_3 - 1)$$

$$+ 2m_3^2\nu_3(\nu_3 - 1)(m_1 + m_2)(m_2 + m_3)L_T(n; 1, 1, \nu_3 + 1)$$

$$- (\nu_3 - 1)(2n - 2\nu_3 - 3)(m_1 + m_2)(m_2 + m_3)L_T(n; 1, 1, \nu_3)$$

$$- (n - \nu_3 - 1)(m_1m_2 + m_1m_3 + m_2m_3)L_T(n; 2, 1, \nu_3 - 1)$$

$$- (n - \nu_3 - 1)m_2^2L_T(n; 1, 2, \nu_3 - 1) + (\nu_3 - 1)m_2^2L(n; 0, 2, \nu_3)$$

$$+ (\nu_3 - 1)(m_1m_2 + m_1m_3 + m_2m_3)L(n; 2, 0, \nu_3), \tag{60}$$

and an analogous equation for $L_T(n; 1, 2, \nu_3)$ (only m_1 and m_2, together with the arguments of the integrals corresponding to ν_1 and ν_2, are to be permuted).

Then, considering equations (32) (which follow from ref. [23]), we obtain the following solution for the remaining integrals (with $\nu_1 = \nu_2 = 1$ and $\nu_1 = \nu_2 = 2$):

$$32m_3^2(\nu_3 - 1)(\nu_3 - 2)(n - \nu_3 - 1)(m_1 + m_2)(m_2 + m_3)(m_3 + m_1)\mu_T L_T(n; 1, 1, \nu_3)$$

$$= 2(\nu_3 - 2)(m_1 + m_2)\Big[(3n - 4\nu_3 - 1)(3n - 4\nu_3 + 1)(m_1 + m_3)(m_2 + m_3)\mu_T$$

$$+ 2(n - 3)(n - \nu_3 - 1)m_3\mu_T(\mu_T + m_3)$$

$$+ (n - 3)(n - 2\nu_3 + 1)m_3(m_1 + m_3)(m_2 + m_3)\Big]L_T(n; 1, 1, \nu_3 - 1)$$

$$- 2(m_1 + m_2)\Big[(n - 2\nu_3 + 1)(m_1 + m_2)(m_2 + m_3)(m_3 + m_1)$$

$$+ 2(n - \nu_3 - 1)m_1m_2(\mu_T + m_3)\Big]L_T(n; 2, 2, \nu_3 - 3)$$

$$+ (m_1 + m_2)\Big\{(n - 3)(m_1 + m_3)(m_2 + m_3)\left[(2n - 11)(m_1 + m_2) - 2m_3\right]$$

$$+ (n - \nu_3 - 1)(n + 2\nu_3 - 9)m_1m_2m_3$$

$$+ (n - \nu_3 - 1)\mu_T\Big[(7n - 20\nu_3 + 39)(m_1 + m_3)(m_2 + m_3)$$

$$- m_3\Big(2(n - 3\nu_3 + 6)(m_1 + m_2) + (3n - 14\nu_3 + 33)m_3\Big)\Big]\Big\}$$

$$\times [L_T(n; 2, 1, \nu_3 - 2) + L_T(n; 1, 2, \nu_3 - 2)]$$

$$+ (n - \nu_3)(m_1 - m_2)\Big\{- (n - 3)(m_1 + m_2)(m_1 + m_3)(m_2 + m_3)$$

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$$+2(n - \nu_3 - 1)\left[\mu_T(2m_1m_2 + 3m_1m_3 + 3m_2m_3 + m_3^2) - m_1m_2m_3\right]\Big\}$$

$$\times\left[L_T(n; 2, 1, \nu_3 - 2) - L_T(n; 1, 2, \nu_3 - 2)\right]$$

$$-2(\nu_3 - 2)(m_1 + m_3)\Big\{(m_1 + m_2)(m_2 + m_3)\left[m_1(n - 5) - m_3(n - 3)\right]$$

$$+2m_3(n - \nu_3 - 1)[\mu_T(m_1 + m_2) + m_1(m_2 + m_3)]\Big\}L(n; 2, 0, \nu_3 - 1)$$

$$-2(\nu_3 - 2)(m_2 + m_3)\Big\{(m_1 + m_2)(m_1 + m_3)\left[m_2(n - 5) - m_3(n - 3)\right]$$

$$+2m_3(n - \nu_3 - 1)[(m_1 + m_2)\mu_T + m_2(m_1 + m_3)]\Big\}L(n; 0, 2, \nu_3 - 1), \qquad (61)$$

$$32(n - \nu_3 - 3)m_1m_2(m_1 + m_2)(m_2 + m_3)(m_3 + m_1)\mu_T L_T(n; 2, 2, \nu_3)$$

$$= 2(n - 3)\nu_3(m_1 + m_2)\Big\{(m_1 + m_3)(m_2 + m_3)\left[(n - 5)(m_1 + m_2) - 2m_3\right]$$

$$-2m_3(n - \nu_3 - 3)\left[\mu_T^2 - m_1m_2\right]\Big\}L_T(n; 1, 1, \nu_3 + 1)$$

$$+2(m_1 + m_2)\Big[(n - 3)(m_1 + m_2)(m_2 + m_3)(m_3 + m_1)$$

$$+2(n - \nu_3 - 3)\mu_T(2m_1m_2 + m_1m_3 + m_2m_3)\Big]L_T(n; 2, 2, \nu_3 - 1)$$

$$+(m_1 + m_2)\Big\{(n - 3)(m_1 + m_3)(m_2 + m_3)\left[(2n - 11)(m_1 + m_2) - 2m_3\right]$$

$$+(n - \nu_3 - 3)(n + 2\nu_3 - 5)m_1m_2m_3$$

$$+(n - \nu_3 - 3)\mu_T\Big[-3(3n - 4\nu_3 - 5)(m_1 + m_3)(m_2 + m_3)$$

$$+m_3\Big(2(n - 3\nu_3)(m_1 + m_2) + 5(n - 2\nu_3 - 1)m_3\Big)\Big]\Big\}$$

$$\times\left[L_T(n; 2, 1, \nu_3) + L_T(n; 1, 2, \nu_3)\right]$$

$$+(n - \nu_3 - 2)(m_1 - m_2)\Big\{-(n - 3)(m_1 + m_2)(m_1 + m_3)(m_2 + m_3)$$

$$+2(n - \nu_3 - 3)\left[-m_1m_2m_3 + \mu_T(2m_1m_2 + m_3\mu_T)\right]\Big\}$$

$$\times\left[L_T(n; 2, 1, \nu_3) - L_T(n; 1, 2, \nu_3)\right]$$

$$+2\nu_3(m_1 + m_3)\Big\{(m_1 + m_2)(m_2 + m_3)\left[(n - 3)m_3 - (n - 5)m_1\right]$$

$$+2m_3(n - \nu_3 - 3)(m_1^2 + m_2\mu_T)\Big\}L(n; 2, 0, \nu_3 + 1)$$

$$+2\nu_3(m_2+m_3)\Big\{(m_1+m_2)(m_1+m_3)\left[(n-3)m_3-(n-5)m_2\right]$$

$$+2m_3(n-\nu_3-3)(m_2^2+m_1\mu_T)\Big\}L(n;0,2,\nu_3+1). \tag{62}$$

We note that the factors $(n-\nu_3-1)$ (on the l.h.s. of eq. (61)) and $(n-\nu_3-3)$ (on the l.h.s. of eq. (62)) yield $(n-4)$ at $\nu_3=3$ and $\nu_3=1$, respectively. This illustrates the problem arising when getting the results for $L_T(n;2,2,1)$ and $L_T(n;1,1,3)$ (see in section 3).

Appendix B: Threshold results for some other integrals

Using eqs. (28), (30) and (41), we obtain

$$L_T(4-2\varepsilon;1,1,3) = \frac{\pi^{4-2\varepsilon}\,\Gamma^2(1+\varepsilon)}{m_3^2}\Bigg\{m_3^{-4\varepsilon}\left[\frac{1}{2\varepsilon}+1-\frac{1}{\mu_T}\left(m_1\ln\frac{m_1}{m_3}+m_2\ln\frac{m_2}{m_3}\right)\right]$$

$$-\frac{(m_1m_2m_3)^{1/2-3\varepsilon}}{2\mu_T^{3/2-5\varepsilon}}\left[\frac{\pi}{2\varepsilon}+4\pi+\pi\mathcal{L}+\mathcal{C}\right]\Bigg\}+\mathcal{O}(\varepsilon). \tag{63}$$

Here and below, $\mathcal{L}\equiv\mathcal{L}(m_1,m_2,m_3)$ and $\mathcal{C}\equiv\mathcal{C}(\theta_1,\theta_2,\theta_3)$, see eqs.(40) and (39).

Employing explicit recurrence relations from Appendix A, we can obtain results for the integrals L_T with higher values of $\sigma=\nu_1+\nu_2+\nu_3$. For instance, for $\sigma=6$ we get

$$L_T(4-2\varepsilon;2,2,2) = \frac{1}{4}\,\pi^{4-2\varepsilon}\,\Gamma^2(1+\varepsilon)\,(m_1m_2m_3)^{-1-3\varepsilon}\,\mu_T^{-1+5\varepsilon}$$

$$\times\Bigg\{-\frac{(m_1+m_2)(m_2+m_3)(m_3+m_1)+4m_1m_2m_3}{4\sqrt{m_1m_2m_3\mu_T^3}}\left[\frac{\pi}{2\varepsilon}+3\pi+\pi\mathcal{L}+\mathcal{C}\right]$$

$$+\pi\sqrt{\frac{m_1m_2m_3}{\mu_T^3}}\left[3+\frac{m_1m_2m_3+\mu_T^3}{(m_1+m_2)(m_2+m_3)(m_3+m_1)}\right]+1$$

$$+\frac{m_3(m_1-m_2)}{(m_2+m_3)(m_3+m_1)}\ln\frac{m_1}{m_2}+\frac{m_1(m_2-m_3)}{(m_3+m_1)(m_1+m_2)}\ln\frac{m_2}{m_3}+\frac{m_2(m_3-m_1)}{(m_1+m_2)(m_2+m_3)}\ln\frac{m_3}{m_1}$$

$$-\frac{1}{2\mu_T}\left[(m_1-m_2)\ln\frac{m_1}{m_2}+(m_2-m_3)\ln\frac{m_2}{m_3}+(m_3-m_1)\ln\frac{m_3}{m_1}\right]\Bigg\}+\mathcal{O}(\varepsilon), \tag{64}$$

$$L_T(4-2\varepsilon;1,2,3) = \frac{1}{8m_2m_3^2}\,\pi^{4-2\varepsilon}\,\Gamma^2(1+\varepsilon)\,(m_1m_2m_3)^{-3\varepsilon}\mu_T^{-1+5\varepsilon}$$

$$\times\Bigg\{\frac{3(m_1+m_2)(m_2+m_3)(m_3+m_1)-4m_1m_3\mu_T}{4\sqrt{m_1m_2m_3\mu_T^3}}\left[\frac{\pi}{2\varepsilon}+\frac{\pi}{3}+\pi\mathcal{L}+\mathcal{C}\right]$$

$$+\frac{\pi\,m_1}{\sqrt{m_1m_2m_3\mu_T}}\left[\frac{2}{3}(3m_2-m_3)-m_2m_3\frac{(m_1+m_2)(m_1+m_3)-m_2^2+m_3^2}{(m_1+m_2)(m_2+m_3)(m_3+m_1)}\right]$$

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$$+1 + \frac{m_1(m_2 - m_3)}{(m_1 + m_2)(m_3 + m_1)} \ln \frac{m_1}{m_2} + \frac{m_3(m_2 - m_1)}{(m_2 + m_3)(m_3 + m_1)} \ln \frac{m_3}{m_2}$$

$$+ \frac{3}{2\mu_T} \left[(m_2 + m_3) \ln \frac{m_1}{m_2} + (m_1 + m_2) \ln \frac{m_3}{m_2} \right] \Bigg\} + \mathcal{O}(\varepsilon). \quad (65)$$

The result for $L_T(4 - 2\varepsilon; 2, 2, 2)$, eq. (64), is totally symmetric in m_1, m_2, m_3, as it should. Using eq. (65) (and its permutations) and the results for lower integrals, one can easily obtain result for $L_T(4 - 2\varepsilon; 1, 1, 4)$ (and its permutations) via eqs. (24)–(25).

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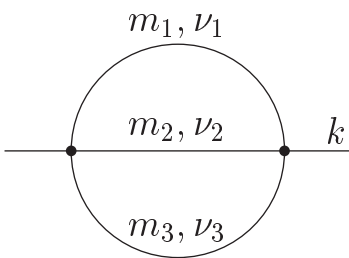E. Byckling and K. Kajantie, *Particle kinematics* (John Wiley, 1973).
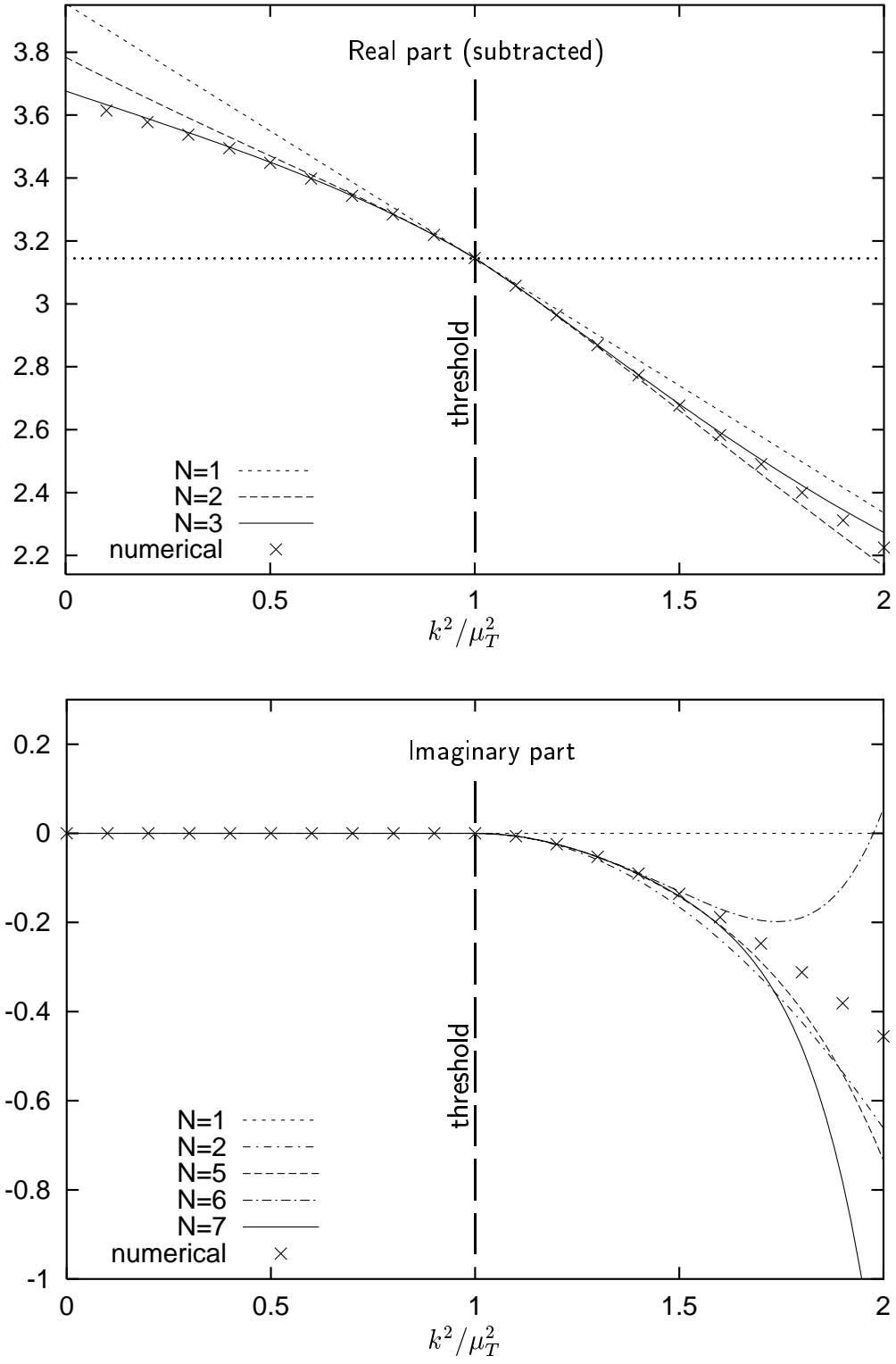
Figure 1 : The sunset diagram

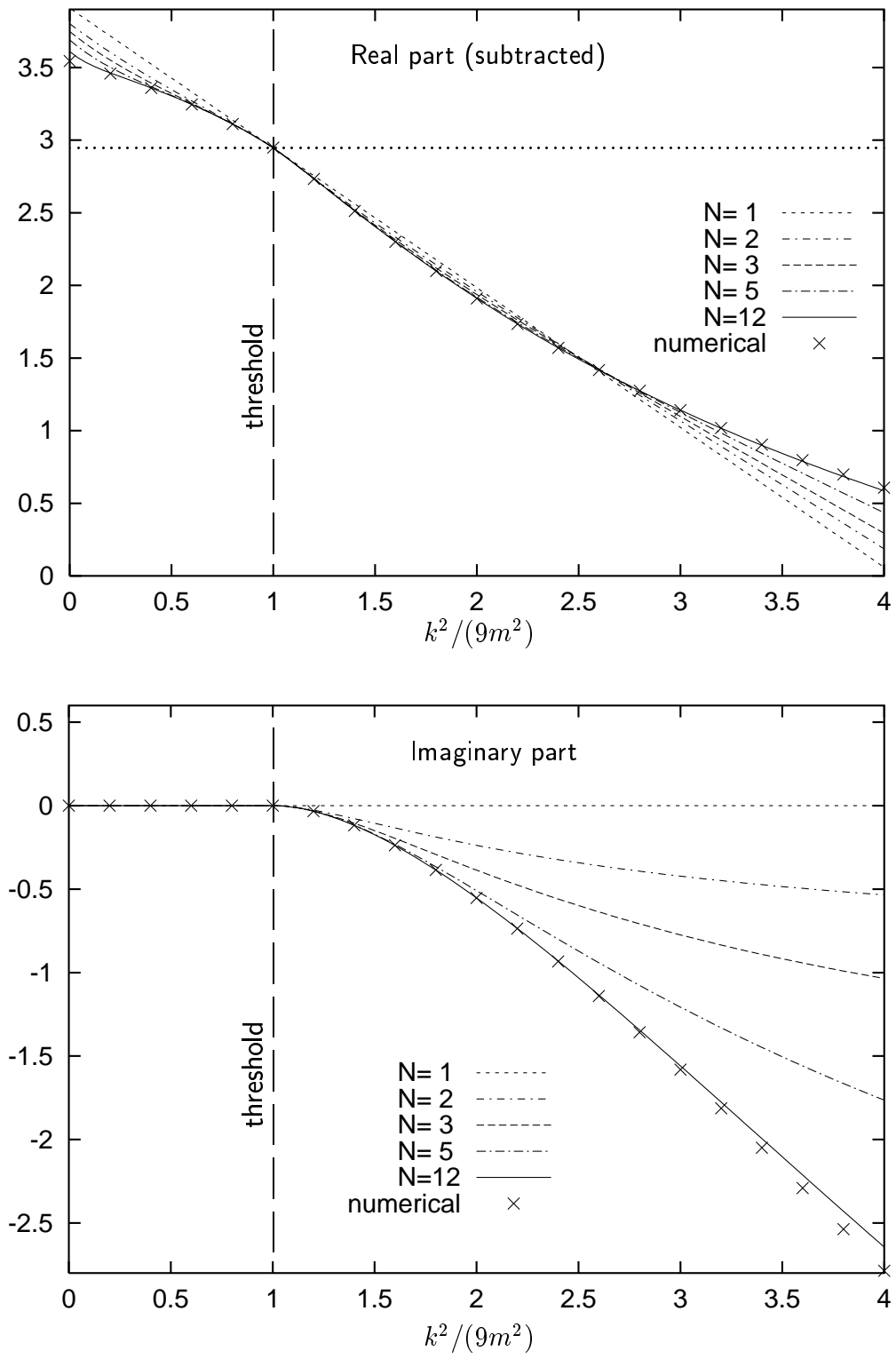Figure 2: Threshold approximations for unequal masses ($\xi_1 = 0.1$, $\xi_2 = 0.3$, $\xi_3 = 0.6$)

Figure 3: Threshold approximations in the case of equal masses